Introductory Euclidean Geometry

by Arnav Kumar

An understanding of similarity, congruence, and several other definitions is assumed.

1 Directed Angles

Please see Evan Chen's handout on the topic (`https://web.evanchen.cc/handouts/Directed-Angles/Directed-Angles.pdf`). There is also the idea of directed line segments and lines, but this is more similar to vectors and need not be covered.

2 Triangles

Theorem 1. (Sum of Interior Angles in a Triangle) *Given a triangle $\triangle ABC$, the sum of the interior angles of the triangle always sum to $180°$ or π radians. In terms of directed angles, we can say that $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0$.*

Corollary 1. *Given a triangle $\triangle ABC$ with D on the extension of AB past B, then we have that $\angle DBC = \angle BCA + \angle CAB$.*

Theorem 2. (Pythagorean Theorem) *Triangle ABC is a right triangle with $\angle ABC = 90°$. Now iff we have $AB^2 + BC^2 = AC^2$.*

Theorem 3. (Area of a Triangle) *Let ABC be a triangle. Let h be the length of the perpendicular dropped from A to BC. Let b be the length of segment BC. The area of ABC, sometimes denoted $|ABC|$ is $\frac{1}{2} b h$.*

There are also some well known triangles which might be useful to know the dimensions of. The most notable of these is the equilateral triangle. If an equilateral triangle has side length 2, then the height is $\sqrt{3}$, and the area is $\sqrt{3}$.

3 Scaling, Transformations, and Translations

While presented informally here, it may be useful to take a look at units to know how much certain values change. For example, if we scale a triangle $3\times$, the area increases $9\times$, because area is length $*$ length and the scaling happens in both directions.

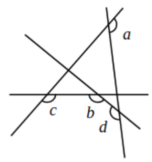
Now we'll get into a very powerful technique in geometry, which is a form of manipulation of our diagram. We can choose translate or rotate parts of our diagram as we wish to form figures which might be conducive for us. There are many examples of this, but here is an example from the *A Taste Of Mathematics* series:

Example 1. In quadrilateral $ABCD$, we have $AD = BC$, and $\angle ABD + \angle BDC = 180°$. Show that $\angle BAD = \angle BCD$.

The trick to this problem is to take the triangle BCD, and transform it (while maintaining angles and lengths) so that the corners on B and D swap positions. Let the position of the last vertex be C'. Now, we see that $180° = \angle ABD + \angle BDC = \angle ABD + \angle DBC'$ which means that B lies on AC'. Additionally, $AD = BC = DC'$ which means that $\triangle ADC'$ is isosceles. Finally, we get $\angle BAD = \angle BC'D = \angle BCD$.

4 Exercises

Exercise 1. (1999 IMC A2) Find the sum of the angles in the following diagram:



Exercise 2. (1999 IMC A4) A 60 meter tall building casts a shadow of length 40 meters. What is the length of the shadow cast by a pole of height 2 meters?

Exercise 3. (1999 IMC A7) Three circles of equal radius, r, are mutually tangent to each other and to the triangle of side length 1 which circumscribes them (and touches each circle twice). Find the value of r.

Exercise 4. (1999 IMC A10) Right triangle ABC has legs AC and AB of length 8 and 15 respectively. Points E and G lie on AC and points D and F lie on AB such that the segments CD, DE, EF, and FG split up ABC into five triangles of equal area. Find which of the segments CD, DE, EF, and FG has integral length.

Exercise 5. (1999 IMC B2) Right triangle ABC has $BC \perp AC$. D lies on BC such that $BC = 4BD$ and E lies on AC such that AC $= 8CE$. Find the length of AB if BE and AD have lengths 52 and 164 respectively.

Exercise 6. (2000 IMC A7) Find $\angle BAD$ if D is a point on BC of $\triangle ABC$ such that $AC = CD$ and $\angle CAB = \angle ABC + 45°$.

Exercise 7. (2000 IMC A9) P is a point in rectangle $ABCD$ such that $PA = 4$, $PB = 6$, and $PD = 9$. Find PC.

IMC refers to the Junior High School Division International Mathematics Competition.