Number Bases and Induction

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1 Number Bases

We often write numbers in base 10, which is to say that the position of any digit in the number represents a power of 10 that is multiplied by the number. Consider $342 = 3 \cdot 10^2 + 4 \cdot 10 + 2$. We can use another number to achieve a different number base.

Definition 1. *Given $n \in \mathbb{Z}$, and a number base b, the* ***base b representation of n****, $\overline{d_k d_{k-1} \ldots d_1 d_0}_{(b)}$, satisfies*

$$n = \sum_{i=0}^{k} d_k \cdot b^k$$

where $0 \leq d_i < b$ and $d_i \in \mathbb{Z}$ for all integers i.

2 Principle of Mathematical Induction

Theorem 1. (Principle of Mathematical Induction) *Let $P(n)$ be a statement that depends on a $n \in \mathbb{Z}^+$. If $P(1)$ is true (called the* ***base case****), and if $(\forall k \in \mathbb{Z}^+)(P(k) \Rightarrow P(k+1))$ (called the* ***inductive step****), then $P(n)$ is true for all positive integers n.*

Exercise 1. Let $n \in \mathbb{Z}^+$. Show that

1. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Theorem 2. (Strong Induction) *Let $P(n)$ be a statement that depends on a $n \in \mathbb{Z}^+$. If $P(1)$ is true, and if $(\forall k \in \mathbb{Z}^+)((P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \Rightarrow P(k+1))$, then $P(n)$ is true for all positive integers n.*

When using induction, it may not be true that what you are trying to prove can be proven by induction. The result which you are trying to prove may be too weak result to give you enough information for the inductive step. In such a case, change the statement to something stronger. For example, consider the following:

Example 1. (AoPS Induction Handout) It is true that for all $n \geq 1$, we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2$$

but proving this is not possible with induction. We have to instead prove a stronger result with induction, that for all $n \geq 1$, we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

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Problem. For a given odd prime number p, define $f(n)$ the remainder of d divided by p, where d is the biggest divisor of n which is not a multiple of p. For example when $p=5$, $f(6)=1$, $f(35)=2$, $f(75)=3$. Define the sequence $a_1, a_2, \ldots, a_n, \ldots$ of integers as the followings:

- $a_1 = 1$
- $a_{n+1} = a_n + (-1)^{f(n)+1}$

Determine all integers m, such that there exist infinitely many positive integers k such that $m = a_k$.

Lemma 1. *If we write n in base p as $n = \overline{c_k c_{k-1} \ldots c_1 c_0 0 \ldots 0}_{(p)}$, where $c_0 \neq 0$, then we see that $f(n) = c_0$.*

Proof. Since we know that $p \nmid d$, and that $d \mid n$, we can write $n = p^e d$. Additionally, there is a q such that $d = pq + f(n)$ where $f(n) < p$ (since it is the remainder when d is divided by p). Taking these together, we see that

$$\begin{aligned} n &= p^e(pq + f(n)) \\ &= p^{e+1} q + p^e f(n) \\ &= p^{e+1} q + f(n)\, p^e + 0 \cdot p^{e-1} + 0 \cdot p^{e-2} + \cdots + 0 \cdot p^1 + 0 \cdot p^0 \end{aligned}$$

Thus, the last non-zero digit of n in base p is $f(n)$. Therefore, $c_0 = f(n)$ as desired. □

Lemma 2. *If $n = \overline{c_k c_{k-1} \ldots c_1 c_0}_{(p)}$, then $a_{n+1} = 1 + \sum_{i=1}^{k} (c_i \mod 2)$. Here, $(x \mod 2)$ is an operation with value 0 or 1.*

Proof. You can observe this by doing examples, and it can be proven with induction.

Base case: $n = 1$

In this base case, we indeed have that $a_2 = a_1 + (-1)^{f(1)+1} = a_1 + 1 = 2 = 1 + (1 \mod 2)$ as desired.

Inductive step: Assuming that if $l = \overline{c_k c_{k-1} \ldots c_1 c_0}_{(p)}$ satisfies $a_{l+1} = 1 + \sum_{i=1}^{l} (c_i \mod 2)$, we will show that $l+1 = \overline{d_k d_{k-1} \ldots d_1 d_0}_{(p)}$ satisfies $a_{l+2} = 1 + \sum_{i=1}^{l+1} (d_i \mod 2)$.

Let j be the smallest integer such that $c_j \neq p-1$. Now, we see that $l+1 = \overline{c_k c_{k-1} \ldots c_{j+1} (c_j+1) 0 \ldots 0}_{(p)}$, thus $d_i = c_i$ if $i > j$, $d_i = c_{i+1}$ if $i = j$, and $d_i = 0$ otherwise. Then,

$$\begin{aligned} a_{l+2} &= a_{l+1} + (-1)^{f(l+1)+1} \\ &= 1 + \left(\sum_{i=0}^{k} (c_i \mod 2) \right) + (-1)^{f(l+1)+1} \\ &= 1 + \left(\sum_{i=j+1}^{k} (c_i \mod 2) \right) + (c_j \mod 2) + (-1)^{f(l+1)+1} \\ &= 1 + \left(\sum_{i=j+1}^{k} (c_i \mod 2) \right) + (c_j \mod 2) + (-1)^{c_j+1+1} \\ &= 1 + \left(\sum_{i=j+1}^{k} (c_i \mod 2) \right) + (c_j + 1 \mod 2) \\ &= 1 + \left(\sum_{i=j+1}^{k} (d_i \mod 2) \right) + (d_j \mod 2) \\ &= 1 + \left(\sum_{i=0}^{k} (d_i \mod 2) \right) \end{aligned}$$

Which is what we wanted to show.

Thus, the induction is complete, and the lemma is proven □

Now, for any $m \in \mathbb{Z}, m > 1$, let $s := \frac{(p^m - 1)}{(p-1)}$. From Lemma 2, we see that for any non-negative i, we have $a_{s \cdot p^i + 1} = m$ because $s \cdot p^i$ has the form $\overline{\underbrace{11\ldots1}_{m\times}\underbrace{00\ldots0}_{i\times}}_{(p)}$.

And when $m = 1$, note that for any non-negative integer i, we have $a_{2p^i+1} = m$ since $2\,p^i = \overline{2\underbrace{00\ldots0}_{i\times}}_{(p)}$.

Thus, $(\forall m \in \mathbb{Z}^+)(\exists \text{ infinitely many } k \in \mathbb{Z}^+)(m = a_k)$.