Points in a Triangle

by Arnav Kumar

Definition 1. *Given a triangle ABC, a* ***cevian*** *is a line segment that connects a vertex to the (possibly extended) opposite side of the triangle.*

1 Circumcircles and Circumcenters

Definition 2. *Given a triangle ABC, we call the unique circle which passes through all 3 points the* ***circumcircle*** *of ABC. By convention, the* ***circumcenter*** *(the center of the circumcircle) is denoted O and has radius equal to the* ***circumradius*** *of R. If $\odot O$ is the circumcircle of ABC, we say that $\triangle ABC$ is* ***inscribed*** *in the circle.*

Corollary 1. *The circumcenter is the meeting point of the perpendicular bisectors of each side of the triangle.*

Theorem 1. *If O is the circumcenter of $\triangle ABC$, then $\angle BOC = 2\,a$, $\angle OBC = \angle OCB = 90° - a$.*

2 Incircles and Incenters

Definition 3. *Given a triangle ABC, we call the unique circle which is tangent to all of the sides of the triangle from the inside the* ***incircle****. The center of the incircle is the* ***incenter*** *and is denoted I and the* ***inradius*** *is denoted r. If $\odot I$ is the incircle of ABC, we say that it is* ***inscribed*** *in $\triangle ABC$.*

Corollary 2. *The incenter is the meeting point of the angle bisectors of each vertex of a triangle.*

Theorem 2. *If I is the incenter of $\triangle ABC$, then $\angle BIC = 90° + \frac{a}{2}$, $\angle IBC = \frac{b}{2}$. and $\angle ICB = \frac{c}{2}$.*

3 Excenters and Excircles

Definition 4. *Given a triangle ABC, we call a circle which is outside the triangle, but tangent to all of the sides of the triangle (or their extensions) an* ***excircle****. The 3 excircles of a triangle correspond to the vertex which is opposite them in the triangle. The center of the A-excircle in triangle ABC is the* ***A-excenter*** *and is denoted I_A with* ***exradius*** *r_A.*

Corollary 3. *The A-excenter is the meeting point of the internal angle bisector at A and the external angle bisectors at B and C.*

Theorem 3. *If I_A is the A-excenter of $\triangle ABC$, then $\angle BI_AC = 90° - \frac{a}{2}$, $\angle AI_AB = \frac{c}{2}$. and $\angle AI_AC = \frac{b}{2}$.*

4 Orthocenters

Definition 5. *Given a triangle ABC, an* ***altitude*** *dropped from any vertex is the cevian from the vertex which is perpendicular to the opposite side of the triangle. The* ***orthocenter*** *is the meeting point of the the 3 altitudes dropped from each vertex to the opposite side and is generally denoted H.*

Theorem 4. *If H is the orthocenter of $\triangle ABC$, then $\angle BHC = 180° - a$, $\angle HBC = 90° - c$. and $\angle HCB = 90° - b$.*

5 Centroids

Definition 6. *Given a triangle ABC, the cevian AD is called a* ***median*** *if $BD = DC$. The 3 medians of a triangle are concurrent and called the* ***centroid****, usually denoted G.*

Theorem 5. *If AD, BE, and CF are medians of triangle ABC which intersect at centroid G, then we have that $AG = 2 \cdot GD$.*

6 Exercises

All of these exercise problems are well known results.

Exercise 1. Show that the circumcenter of a right triangle is the midpoint of its hypotenuse.

Exercise 2. Show that given a triangle ABC, the intersections of the internal and external bisectors of $\angle BAC$ with the perpendicular bisector of BC both lie on the circumcircle of ABC.

Exercise 3. Let ABC be a triangle with circumcenter O and orthocenter H. Let D be the point directly opposite A on the circumcircle, let M be the midpoint of BC, and let E be the reflection of H on BC. Show that

1. $BHCD$ is a parallelogram
2. M is the midpoint of HD
3. E lies on the circumcircle of ABC
4. $\angle BAH = \angle CAO$