Cyclic Quadrilaterals and More Triangles

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1 Cyclic Quadrilaterals

Definition 1. *We call a quadrilateral which can be incribed in a circle a **cyclic quadrilateral** (or a **concyclic quadrilateral**) and we say that the circle circumscribes the quadrilateral.*

From the previous lesson about points in a triangle, we learnt about the properties of triangles inscribed in a circle, and because we know that there are $\binom{4}{3} = 4$ distinct triangles in a cyclic quadrilateral, we can make some statements with our knowledge of angles while dealing with circumcircles.

Theorem 1. *Any 4 points A, B, C, and D are concyclic iff $\measuredangle ABC = \measuredangle ADC$.*

2 Related Theorems

Theorem 2. (Power of a Point) *If $ABCD$ is a convex quadrilateral with AB and CD intersecting at P and AC and BD intersecting at Q, then $ABCD$ is cyclic iff either:*

1. $AQ \cdot QC = BQ \cdot QD$ *(or equivalently $QAD \sim QBC$)*
2. $PA \cdot PB = PC \cdot PD$ *(or equivalently $PAD \sim PCB$)*

Theorem 3. (Ptolemy's Theorem) *Quadrilateral $ABCD$ is cyclic iff*

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

Theorem 4. (Brahmagupta's Formula) *If $ABCD$ is a cyclic quadrilateral with sides of length a, b, c, and d, then let the semiperimeter be $s := \frac{1}{2}(a + b + c + d)$. We have that:*

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Corollary 1. (Heron's Formula) *For a triangle ABC, if s is the semiperimeter, $\frac{1}{2}(a + b + c)$, we have:*

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

3 Exercises

Exercise 1. Let ABC be a triangle with heights AD, BE, and CF. Show that:

1. The incenter of $\triangle DEF$ is the orthocenter of $\triangle ABC$.
2. $\angle DEF = 180° - 2\angle ABC$

Exercise 2. This is a well known fact. Prove that if l is a line tangent to the circumcircle of ABC at A, and D lies on l such that $\angle DAC > \angle DAB$, then $\angle DAB = \angle ACB$.

Exercise 3. (2022 KJMO Q1) The inscribed circle of an acute triangle ABC meets the segments AB and BC at D and E respectively. Let I be the incenter of the triangle ABC. Prove that the intersection of the line AI and DE is on the circle whose diameter is AC.

Exercise 4. Let ABC be an equilateral triangle, and let P lie on the minor arc BC of its circumcircle. Show $PA = PB + PC$.

Exercise 5. Let $ABCDEFG$ be a regular heptagon. Show that $\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AE}$.

KJMO refers to the Korean Junior Math Olympiad.